Fluctuation-dissipation-relation-preserving field theory of the glass transition in terms of fluctuating hydrodynamics

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A field theoretical method for fluctuating hydrodynamics with preserved fluctuation-dissipation relations is reformulated. By assuming that the correlations including momentum are irrelevant in the long time region, we demonstrate that the equation obtained from the first-order perturbation is reduced to that for standard mode-coupling theory.

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I. INTRODUCTION

In the vicinity of the glass transition point, the dynamics of supercooled liquids becomes extremely slow [1–3]. The dynamics of glass transition has attracted considerable attention over the years. Among the many theoretical approaches employed to study the dynamics, the mode-coupling theory (MCT) is one of the most successful ones that can be "derived" from first principles, and it explains many aspects of observations in experiments and simulations, such as multistep relaxation processes and the Debye-Waller parameter [4–7].

Despite these advantages of the MCT, some controversial aspects remain regarding the validity of the standard MCT (SMCT). Indeed, the SMCT predicts the existence of ergodic-nonergodic (ENE) transition, where the system becomes nonergodic below a critical temperature or above a critical density, while real systems are still ergodic in experiments and simulations at low temperature or high density. Furthermore, the SMCT predicts an algebraic divergence of the viscosity at the critical point of ENE transition, but the viscosity for real supercooled liquids obeys the Vogel-Fulcher law near the glass transition point. Moreover, the Vogel-Fulcher temperature is lower than the critical temperature of ENE transition. To overcome these difficulties of the SMCT, many investigations have been carried out [8–29]. It may be concluded that the failures of the SMCT originated from the decoupling approximation of a four-point correlation function. In fact, Mayer *et al.* [21] introduced a toy model that does not have any spatial degrees of freedom, and they demonstrated that the ergodicity of the system at a low temperature is recovered when higher-order correlations are included. On the other hand, there exists ENE transition within the framework of the decoupling approximation. This suggests that we should not adopt the decoupling approximation but rather, use an approximation that contains higher-order correlations. However, the systematic improvement of the approximation by using the projection operator technique is difficult within the conventional framework.

The field theoretical approach is a promising method that can systematically improve approximations. Another advantage of the field theory regards the response function and the fluctuation-dissipation relations (FDR). Following the Martin-Siggia-Rose (MSR) method [30], we can construct an action by the introduction of conjugate fields, for a set of nonlinear Langevin equations, and use perturbative expansion. Currently, however, there are points of confusion related to the use of this approach. Indeed, among many field theoretical investigations [8,10,12,20,23,27–29], only a few papers have reported successful derivation of the SMCT in lowest order perturbation from nonlinear Langevin equations. One of the main confusing areas is the violation of FDR in each order of naive perturbative expansions of the set of nonlinear Langevin equations, as indicated by Miyazaki and Reichman [20].

In order to recover FDR preservation at each order of perturbation, recently, Andreanov, Biroli, and Lefèvre (ABL) [23] indicated the importance of the time-reversal symmetry of action, and they introduced some additional field variables. Indeed, ABL demonstrated that we can construct a FDR-preserving field theory, starting from the nonlinear Langevin equations that contain both the Dean-Kawasaki equation and the fluctuating nonlinear hydrodynamic (FNH) equations. Kim and Kawasaki [28] further improved the ABL method and derived a mode coupling equation, similar to the SMCT, from the Dean-Kawasaki equation [11,31] in the first-loop order via the irreducible memory functional approach. This approach is essential for treating the dynamics of dissipative systems such as the interacting Brownian particle system.

On the other hand, Das and Mazenko [8] presented a pioneering paper on the field theoretical approach of FNH. They suggested the existence of a cutoff mechanism in which the system is always ergodic, even at a low temperature. Later, Schmitz, Dufty, and De (SDD) [10] arrived at the same conclusion through a concise discussion, however, they destroyed the Galilean invariance of FNH equations. On the other hand, Kawasaki [11] suggested that FNH equations reduce to the Dean-Kawasaki equation in the long time limit. Furthermore, ABL [23] suggested the existence of ENE transition in FNH, and indicated that the calculation by Das and Mazenko breaks FDR preservation. Moreover, Cates and Ramaswamy [22] indicated that the calculation by Das and Mazenko violates the momentum conservation. Das and Mazenko [32], however, responded that the indications by ABL and those by Cates and Ramaswamy do not imply fatal er-

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rors on the part of Das and Mazenko [8] but rather, reveal the existence of some misleading arguments. Thus, there continue to be points of confusion regarding the use of the field theory in glass transition, and it remains to be concluded whether ENE transition exists in FNH.

In this paper, we apply the method developed by Kim and Kawasaki [28] to FNH to clarify the current situation with regard to the application of the FDR-preserving field theoretical approach to glass transition. The organization of this paper is as follows. In the next section, we introduce FNH, which describes the time evolutions of the density field and the momentum field, agitated by a fluctuating random force for compressible fluids. This set of equations is equivalent to that used by Das and Mazenko [8] and ABL [23]. In the first half of Sec. III, we create an action invariant under the time-reversal transformation. In order to maintain the linearity of the time-reversal transformation, we introduce some additional variables and their conjugate fields. This linearity of the time-reversal transformation makes FDR-preserving field theory possible. We also introduce a complete set of Schwinger-Dyson equations of our problem, and we summarize some identities used for perturbative calculation in the latter half of Sec. III. Section IV is the main part of our paper, where we explain the detailed calculations of perturbative expansion within the first-loop order. The calculations are done under the assumption that the correlations including momentum can be ignored in the long time limit. Within this approximation, we predict the existence of ENE transition, and we obtain an equation equivalent to that obtained by the SMCT. In the last section, we discuss the validity of our assumptions and compare our results with others. We also summarize our results. In Appendix A, we present the details of the time-reversal transformation and provide some relevant relations derived from the time-reversal transformation. In Appendix B, we present the details of the calculation for one component of the Schwinger-Dyson equation. In Appendix C, we show some relations for the equal-time correlations and self-energies. In Appendix D, we present explicit expressions for all three-point vertex functions.

II. FLUCTUATING NONLINEAR HYDRODYNAMICS

In this section, we briefly summarize our basic equations, FNH, and MSR action [30]. The argument in this section is parallel to those presented in the previous studies [8,23].

Let us describe a system of supercooled liquids in terms of a set of equations for the density field $\rho(\boldsymbol{r},t)$ and the momentum field $\boldsymbol{g}(\boldsymbol{r},t)$. For the continuity equation of momentum, we employ the Navier-Stokes equation for compressible fluids supplemented with the osmotic pressure induced by the density fluctuation and the noise caused by the fast fluctuations. In order to keep the analysis simple, we ignore the fluctuations of energy [33] as assumed by Das and Mazenko [8], SDD [10], and ABL [23].

The time evolutions of the collective variables ρ and $\boldsymbol{g}$, which we call FNH equations, are given by [8,23]

$$\partial_t \rho = -\nabla \cdot \boldsymbol{g}, \tag{1}$$

$$\partial_t g_\alpha = -\rho \nabla_\alpha \frac{\delta F_U}{\delta \rho} - \nabla_\beta \frac{g_\alpha g_\beta}{\rho} - L_{\alpha\beta}\frac{g_\beta}{\rho} + \eta_\alpha. \tag{2}$$

Here, η_α is the Gaussian white noise with zero mean, which satisfies

$$\langle \eta_\alpha(\boldsymbol{r},t)\eta_\beta(\boldsymbol{r}',t')\rangle = 2TL_{\alpha\beta}\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t'), \tag{3}$$

where T is temperature and $L_{\alpha\beta}$ is the operator tensor acting on any field variables $h(\boldsymbol{r})$ as

$$L_{\alpha\beta}h(\boldsymbol{r}) = -[\mu_0(\tfrac{1}{3}\nabla_\alpha\nabla_\beta + \delta_{\alpha\beta}\nabla^2) + \zeta_0\nabla_\alpha\nabla_\beta]h(\boldsymbol{r}), \tag{4}$$

with shear viscosity μ_0 and bulk viscosity ζ_0. Note that hereafter, the Boltzmann's constant is set to unity. In this paper, greek indices such as α are used for the spatial components, and Einstein's rule $g_\alpha g_\alpha \equiv \Sigma_{\alpha=1}^3 g_\alpha^2$ is adopted. The effective free-energy functional $F=F_K+F_U$ consists of the kinetic part F_K and the potential part F_U as

$$F_K = \frac{1}{2}\int d\boldsymbol{r}\frac{\boldsymbol{g}^2(\boldsymbol{r})}{\rho(\boldsymbol{r})}, \tag{5}$$

$$F_U = \frac{T}{m}\int d\boldsymbol{r}\rho(\boldsymbol{r})\left[\ln\left(\frac{\rho(\boldsymbol{r})}{\rho_0}\right)-1\right] - \frac{T}{2m^2}\int d\boldsymbol{r}d\boldsymbol{r}'c(\boldsymbol{r}-\boldsymbol{r}')\delta\rho(\boldsymbol{r})\delta\rho(\boldsymbol{r}'), \tag{6}$$

where m is the mass of a particle and $c(\boldsymbol{r})$ is the direct correlation function [34]. The potential part F_U of the effective free-energy functional has the same form as the Ramakrishnan-Yussouff form [35]. Here, $\delta\rho(\boldsymbol{r},t)\equiv\rho(\boldsymbol{r},t)-\rho_0$ is the local density fluctuation around the mean density ρ_0. From relations (5) and (6), we can rewrite (1) and (2) as

$$\partial_t\rho = -\nabla\left(\rho\frac{\delta F}{\delta \boldsymbol{g}}\right), \tag{7}$$

$$\partial_t g_\alpha = -\rho\nabla_\alpha\frac{\delta F}{\delta\rho} - \nabla_\beta\left(g_\alpha\frac{\delta F}{\delta g_\beta}\right) - g_\beta\nabla_\alpha\frac{\delta F}{\delta g_\beta} - L_{\alpha\beta}\frac{\delta F}{\delta g_\beta} + \eta_\alpha, \tag{8}$$

where we have used $\rho\nabla_\alpha(\delta F_K/\delta\rho) = -g_\beta\nabla_\alpha(\delta F/\delta g_\beta)$.

In general, it is impossible to solve the set of nonlinear partial differential equations (1)–(6). In this paper, we adopt the MSR field theory [30]. Let us derive the MSR action. Because the collective variables ρ and $\boldsymbol{g}$ satisfy the dynamic equations (7) and (8), the average of an observable $A[\rho,\boldsymbol{g}]$ is expressed as

$$\begin{aligned}\langle A\rangle &= \left\langle \int D\rho' D\boldsymbol{g}' A[\rho',\boldsymbol{g}']\delta(\rho'-\rho)\delta(\boldsymbol{g}'-\boldsymbol{g})\right\rangle \\ &= \int D\rho D\boldsymbol{g} J(\rho,\boldsymbol{g}) A[\rho,\boldsymbol{g}]\left\langle \delta\left[\partial_t\rho + \nabla\left(\rho\frac{\delta F}{\delta\boldsymbol{g}}\right)\right]\right. \\ &\quad \times \prod_\alpha \delta\left[\partial_t g_\alpha + \rho\nabla_\alpha\frac{\delta F}{\delta\rho} + \nabla_\beta\left(g_\alpha\frac{\delta F}{\delta g_\beta}\right) + g_\beta\nabla_\alpha\frac{\delta F}{\delta g_\beta}\right. \\ &\quad \left.\left. + L_{\alpha\beta}\frac{\delta F}{\delta g_\beta} - \eta_\alpha\right]\right\rangle,\end{aligned} \tag{9}$$

where $J(\rho,\boldsymbol{g})$ is the Jacobian. As written in Ref. [36], the Jacobian $J(\rho,\boldsymbol{g})$ can be independent of both ρ and $\boldsymbol{g}$ when we employ the Itô interpretation. When we replace the δ functions by the functional integrals of the conjugate fields $\hat{\rho}$ and $\hat{g}_\alpha$, the average of A in Eq. (9) can be rewritten as

$$\langle A\rangle = \frac{1}{Z_0}\int D\rho D\boldsymbol{g} D\hat{\rho} D\hat{\boldsymbol{g}} A[\rho,\boldsymbol{g}] \times \left\langle \exp\left(\int d\boldsymbol{r}dt\left\{-\hat{\rho}\left[\partial_t\rho + \nabla\left(\rho\frac{\delta F}{\delta \boldsymbol{g}}\right)\right] - \hat{g}_\alpha\left[\partial_t g_\alpha + \rho\nabla_\alpha\frac{\delta F}{\delta\rho} + \nabla_\beta\left(g_\alpha\frac{\delta F}{\delta g_\beta}\right) + g_\beta\nabla_\alpha\frac{\delta F}{\delta g_\beta} + L_{\alpha\beta}\frac{\delta F}{\delta g_\beta} - \eta_\alpha\right]\right\}\right)\right\rangle, \tag{10}$$

where Z_0 is the normalization constant. By means of Eq. (3), the average of A is given by

$$\langle A\rangle = \frac{1}{Z_0}\int D\rho D\boldsymbol{g} D\hat{\rho} D\hat{\boldsymbol{g}} A[\rho,\boldsymbol{g}] \times \exp\left(\int d\boldsymbol{r}dt\left\{-\hat{\rho}\left[\partial_t\rho + \nabla\left(\rho\frac{\delta F}{\delta \boldsymbol{g}}\right)\right] - \hat{g}_\alpha\left[\partial_t g_\alpha + \rho\nabla_\alpha\frac{\delta F}{\delta\rho} + \nabla_\beta\left(g_\alpha\frac{\delta F}{\delta g_\beta}\right) + g_\beta\nabla_\alpha\frac{\delta F}{\delta g_\beta} + L_{\alpha\beta}\frac{\delta F}{\delta g_\beta}\right]\right\}\right) \times \left\langle \exp\left(\int d\boldsymbol{r}dt\,\hat{g}_\alpha\eta_\alpha\right)\right\rangle = \frac{1}{Z_0}\int D\rho D\boldsymbol{g} D\hat{\rho} D\hat{\boldsymbol{g}} A[\rho,\boldsymbol{g}] e^{S[\rho,\hat{\rho},\boldsymbol{g},\hat{\boldsymbol{g}}]}, \tag{11}$$

where the MSR action $S[\rho,\hat{\rho},\boldsymbol{g},\hat{\boldsymbol{g}}]$ is defined by

$$S[\rho,\hat{\rho},\boldsymbol{g},\hat{\boldsymbol{g}}] \equiv \int d\boldsymbol{r}dt\left\{-\hat{\rho}\left[\partial_t\rho + \nabla_\alpha\left(\rho\frac{\delta F}{\delta g_\alpha}\right)\right] + T\hat{g}_\alpha L_{\alpha\beta}\hat{g}_\beta - \hat{g}_\alpha\left[\partial_t g_\alpha + \rho\nabla_\alpha\frac{\delta F}{\delta\rho} + \nabla_\beta\left(g_\alpha\frac{\delta F}{\delta g_\beta}\right) + g_\beta\nabla_\alpha\frac{\delta F}{\delta g_\beta} + L_{\alpha\beta}\frac{\delta F}{\delta g_\beta}\right]\right\}. \tag{12}$$

III. CONSTRUCTION OF FDR-PRESERVING FIELD THEORY

A. Time-reversal symmetry in action

In order to construct a FDR-preserving field theory, it is necessary to introduce some new variables and the linear time-reversal transformation, which makes the MSR action invariant. It is easy to check whether the action (12) is invariant under the time-reversal transformation [23]

$$t\to -t,\quad \rho\to\rho,\quad \hat{\rho}\to -\hat{\rho} + \frac{1}{T}\frac{\delta F}{\delta\rho},$$

$$g_\alpha \to -g_\alpha,\quad \hat{g}_\alpha\to\hat{g}_\alpha - \frac{1}{T}\frac{\delta F}{\delta g_\alpha}. \tag{13}$$

Here, we adopt the method developed by Kim and Kawasaki [28], in which they introduced the new variable

$$\theta_{KK} \equiv \frac{\delta F_U}{\delta\rho} - A*\delta\rho, \tag{14}$$

where $A*\delta\rho$ represents the linear part of $\delta F_U/\delta\rho$ on $\delta\rho$. Kim and Kawasaki [28] confirmed that the density correlation function of the Dean-Kawasaki equation for the noninteracting case satisfies the diffusion equation nonperturbatively. Furthermore, they concluded that the nonergodic parameter is the same as that of the SMCT.

Following the idea of Kim and Kawasaki [28], to eliminate the nonlinearity of the time-reversal transformation of (13), we introduce new variables θ and $\boldsymbol{\nu}$,

$$\theta \equiv \frac{1}{T}\frac{\delta F}{\delta\rho} - K*\delta\rho, \tag{15}$$

$$\nu_\alpha \equiv \frac{1}{T}\frac{\delta F}{\delta g_\alpha} - \frac{1}{T\rho_0}g_\alpha, \tag{16}$$

where the operator K acts on any function h as

$$K*h(\boldsymbol{r}) \equiv \frac{1}{m\rho_0}\int d\boldsymbol{r}'\left(\delta(\boldsymbol{r}-\boldsymbol{r}') - \frac{\rho_0}{m}c(\boldsymbol{r}-\boldsymbol{r}')\right)h(\boldsymbol{r}'). \tag{17}$$

It should be noted that the right-hand sides (RHS) of Eqs. (15) and (16) do not include the zeroth and first order of $\delta\rho$ and $\boldsymbol{g}$. The choices of Eqs. (15) and (16) differ from those used by ABL [23]. The implication of the difference will be discussed in Sec. V.

As a result of the introduction of θ and ν_α, action (12) can be rewritten as

$$S[\psi] = \int d\boldsymbol{r}dt\big(-\hat{\rho}\{\partial_t\rho + \nabla_\alpha[\rho(\rho_0^{-1}g_\alpha + T\nu_\alpha)]\} - \hat{g}_\alpha\{\partial_t g_\alpha + T\rho\nabla_\alpha(K*\delta\rho+\theta) + L_{\alpha\beta}(\rho_0^{-1}g_\beta + T\nu_\beta) + \nabla_\beta[g_\alpha(\rho_0^{-1}g_\beta + T\nu_\beta)] + g_\beta\nabla_\alpha(\rho_0^{-1}g_\beta + T\nu_\beta)\} + T\hat{g}_\alpha L_{\alpha\beta}\hat{g}_\beta - \hat{\theta}(\theta - f_\theta) - \hat{\nu}_\alpha(\nu_\alpha - f_{\nu_\alpha})\big), \tag{18}$$

where we have introduced

$$f_\theta(\delta\rho,\boldsymbol{g}) \equiv \frac{1}{T}\frac{\delta F}{\delta\rho} - K*\delta\rho, \tag{19}$$

$$f_{\nu_\alpha}(\delta\rho,\boldsymbol{g}) \equiv \frac{1}{T}\frac{\delta F}{\delta g_\alpha} - \frac{1}{T\rho_0}g_\alpha. \tag{20}$$

We have also used the abbreviation of a set of the field variables $\psi^T \equiv (\delta\rho,\hat{\rho},\theta,\hat{\theta},\boldsymbol{g},\hat{\boldsymbol{g}},\boldsymbol{\nu},\hat{\boldsymbol{\nu}})$. Here, the time-reversal transformation, which makes action (18) invariant, is given by

$$t\to -t,\quad \rho\to\rho,\quad \hat{\rho}\to -\hat{\rho} + \theta + K*\delta\rho,$$

$$g_\alpha \to -g_\alpha, \quad \hat{g}_\alpha \to \hat{g}_\alpha - \nu_\alpha - \frac{1}{T\rho_0} g_\alpha, \quad \theta \to \theta,$$

$$\hat{\theta} \to \hat{\theta} + \partial_t \rho, \quad \nu_\alpha \to -\nu_\alpha, \quad \hat{\nu}_\alpha \to -\hat{\nu}_\alpha - \partial_t g_\alpha. \quad (21)$$

We can thus construct a FDR-preserving field theory, due to the linearity of the time-reversal transformation (21). As in typical cases, let us split action (18) into the Gaussian part S_g, which represents the bilinear terms of the field variables, and the non-Gaussian part S_{ng} as

$$S[\psi] = S_g[\psi] + S_{ng}[\psi], \quad (22)$$

where

$$S_g[\psi] = \int d\boldsymbol{r}dt\{-\hat{\rho}(\partial_t \rho + \nabla_\alpha g_\alpha + \underline{T\rho_0 \nabla_\alpha \nu_\alpha}) - \hat{g}_\alpha[\partial_t g_\alpha + T\rho_0 \nabla_\alpha K * \delta\rho + T\rho_0 \nabla_\alpha \theta + L_{\alpha\beta}(\rho_0^{-1} g_\beta + T\nu_\beta)] + T\hat{g}_\alpha L_{\alpha\beta} \hat{g}_\beta - \hat{\theta}\theta - \hat{\nu}_\alpha \nu_\alpha\}, \quad (23)$$

and

$$S_{ng}[\psi] = \int d\boldsymbol{r}dt\{-\hat{\rho}\{\underline{\nabla_\alpha[\delta\rho(\rho_0^{-1} g_\alpha + T\nu_\alpha)]}\} - \hat{g}_\alpha\{T\delta\rho\nabla_\alpha(K * \delta\rho + \theta) + \nabla_\beta[g_\alpha(\rho_0^{-1} g_\beta + T\nu_\beta)] + g_\beta \nabla_\alpha(\rho_0^{-1} g_\beta + T\nu_\beta)\} + \hat{\theta} f_\theta(\delta\rho, \boldsymbol{g}) + \hat{\nu}_\alpha f_{\nu_\alpha}(\delta\rho, \boldsymbol{g})\}. \quad (24)$$

Note that we present some relations in the time-reversal symmetry of this action in Appendix A.

It should be noted that continuity equation (7) can be rewritten as

$$\partial_t \rho = -\nabla_\alpha[\rho(T\nu_\alpha + \rho_0^{-1} g_\alpha)] = -\nabla \cdot \boldsymbol{g} - T\rho_0 \nabla_\alpha \cdot \nu_\alpha - \nabla_\alpha[\delta\rho(T\nu_\alpha + \rho_0^{-1} g_\alpha)], \quad (25)$$

where we have used Eq. (16). From Eqs. (1) and (25), we obtain the identity

$$T\rho_0 \nabla_\alpha \nu_\alpha + \nabla_\alpha[\delta\rho(T\nu_\alpha + \rho_0^{-1} g_\alpha)] = 0. \quad (26)$$

Therefore, the sum of the underlined terms in Eqs. (23) and (24) should be zero. However, each underlined term is separately included in the Gaussian part (23) or the non-Gaussian part (24). To satisfy the action invariant under the time-reversal transformation in each part, we should retain these terms in the calculation.

B. Exact results of Schwinger-Dyson equation

In this section, we derive a set of closed equations of the two-point correlation function. Let us express the two-point correlation function in the matrix form as

$$\mathsf{G}(\boldsymbol{r} - \boldsymbol{r}', t - t') \equiv \langle \psi(\boldsymbol{r}, t) \psi^T(\boldsymbol{r}', t') \rangle, \quad (27)$$

and its $\psi\psi'$ component is represented by

$$G_{\psi\psi'}(\boldsymbol{r} - \boldsymbol{r}', t - t') \equiv \langle \psi(\boldsymbol{r}, t) \psi'(\boldsymbol{r}', t') \rangle, \quad (28)$$

where ψ or ψ' is one of the components of ψ. Note that hereafter, we adopt the simple notation $\psi = \rho$ to represent the contribution from $\delta\rho$. With the aid of the Fourier transform of $h(\boldsymbol{r})$

$$h(\boldsymbol{r}) = \int \frac{d\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} h(\boldsymbol{k}), \quad (29)$$

and action (22), we obtain the Schwinger-Dyson (SD) equation

$$[\mathsf{G}_0^{-1} \cdot \mathsf{G}](\boldsymbol{k}, t) - [\Sigma \cdot \mathsf{G}](\boldsymbol{k}, t) = \mathsf{I}\delta(t), \quad (30)$$

where Σ and I are, respectively, the self-energy matrix and the unit matrix. Here, the free propagator matrix G_0 satisfies

$$S_g[\psi] = -\frac{1}{2} \int dX_1 dX_2 \psi^T(X_1) \mathsf{G}_0^{-1}(X_1 - X_2) \psi(X_2), \quad (31)$$

where we have used the abbreviations as $X_i \equiv (\boldsymbol{r}_i, t_i)$ with $i = 1, 2$. We note that the SD equation (30) is an equation for 16×16 matrices.

Here, we indicate that $G_{\rho\rho}(\boldsymbol{k}, 0)$ is related to the static structure factor $S(\boldsymbol{k})$ as

$$S(\boldsymbol{k}) \equiv \frac{1}{m\rho_0} G_{\rho\rho}(\boldsymbol{k}, 0). \quad (32)$$

From the definition of the direct correlation function [34], Eq. (32) can be rewritten as

$$S(\boldsymbol{k}) = \left(1 - \frac{\rho_0}{m} c(\boldsymbol{k})\right)^{-1} = \frac{1}{m\rho_0} K^{-1}(\boldsymbol{k}). \quad (33)$$

Let us explicitly write some components of the SD equation

$$\partial_t G_{\rho\phi}(\boldsymbol{k}, t) + ik_\alpha G_{g_\alpha \phi}(\boldsymbol{k}, t) + iT\rho_0 k_\alpha G_{\nu_\alpha \phi}(\boldsymbol{k}, t) = F_{\hat{\rho}\phi}(\boldsymbol{k}, t), \quad (34)$$

$$G_{\theta\phi}(\boldsymbol{k}, t) + \Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k}, 0) G_{\rho\phi}(\boldsymbol{k}, t) = F_{\hat{\theta}\phi}(\boldsymbol{k}, t), \quad (35)$$

$$\partial_t G_{g_\alpha \phi}(\boldsymbol{k}, t) + TL_{\alpha\beta}\left(\frac{1}{T\rho_0} G_{g_\beta \phi}(\boldsymbol{k}, t) + G_{\nu_\beta \phi}(\boldsymbol{k}, t)\right) + iT\rho_0 k_\alpha [K(\boldsymbol{k}) G_{\rho\phi}(\boldsymbol{k}, t) + G_{\theta\phi}(\boldsymbol{k}, t)] = F_{\hat{g}_\alpha \phi}(\boldsymbol{k}, t), \quad (36)$$

$$G_{\nu_\alpha \phi}(\boldsymbol{k}, t) + \Sigma_{\hat{\nu}_\alpha \hat{\nu}_\beta}(\boldsymbol{k}, 0) G_{g_\beta \phi}(\boldsymbol{k}, t) = F_{\hat{\nu}_\alpha \phi}(\boldsymbol{k}, t), \quad (37)$$

where ϕ is one of the components of $\phi^T \equiv (\delta\rho, \theta, \boldsymbol{g}, \boldsymbol{\nu})$. Here, Eqs. (34)–(37) are obtained from $\hat{\rho}\phi$, $\hat{\theta}\phi$, $\hat{g}_\alpha \phi$, and $\hat{\nu}_\alpha \phi$ components of the SD equation, respectively. The derivation of Eq. (35) is shown in Appendix B as one example. The derivation of the other equations is parallel to that for Eq. (35). We express the RHS of Eqs. (34)–(37) as a unified form $F_{\hat{\phi}\phi}$ where $\hat{\phi}$ is one of the components of $\hat{\phi}^T \equiv (\hat{\rho}, \hat{\theta}, \hat{\boldsymbol{g}}, \hat{\boldsymbol{\nu}})$. From a parallel argument to derive $F_{\hat{\theta}\phi}$, i.e., the RHS of Eq. (B5), $F_{\hat{\phi}\phi}$ satisfies

$$F_{\hat{\phi}\phi}(\boldsymbol{k},t) = \int_0^t ds\Big[-\Sigma_{\hat{\phi}\hat{\rho}}(\boldsymbol{k},t-s)[K(\boldsymbol{k})G_{\rho\phi}(\boldsymbol{k},s) + G_{\theta\phi}(\boldsymbol{k},s)] + \Sigma_{\hat{\phi}\hat{\theta}}(\boldsymbol{k},t-s)\partial_s G_{\rho\phi}(\boldsymbol{k},s) - \Sigma_{\hat{\phi}\hat{g}_\alpha}(\boldsymbol{k},t-s) \times\left(\frac{1}{T\rho_0}G_{g_\alpha\phi}(\boldsymbol{k},s) + G_{\nu_\alpha\phi}(\boldsymbol{k},s)\right) + \Sigma_{\hat{\phi}\hat{\nu}_\alpha}(\boldsymbol{k},t-s)\partial_s G_{g_\alpha\phi}(\boldsymbol{k},s)\Big]. \quad (38)$$

These components of the SD equation are so complicated because of the self-energies. However, we can simplify these components with the aid of some exact relations. First, from Eqs. (C4) and (C8), Eqs. (35) and (37) are, respectively, simplified as

$$G_{\theta\phi}(\boldsymbol{k},t) = F_{\hat{\theta}\phi}(\boldsymbol{k},t), \quad (39)$$

$$G_{\nu_\alpha\phi}(\boldsymbol{k},t) = F_{\hat{\nu}_\alpha\phi}(\boldsymbol{k},t). \quad (40)$$

Second, from Eqs. (23) and (24), the following identity is obtained:

$$\left\langle \frac{\delta S[\psi]}{\delta\hat{\rho}(\boldsymbol{r},t)}\phi(\boldsymbol{r}',t')\right\rangle = 0. \quad (41)$$

This identity can be expressed by the explicit form

$$\langle\{\partial_t\rho + \nabla\cdot\boldsymbol{g} + T\rho_0\nabla\cdot\boldsymbol{\nu}\cdot[\delta\rho(T\boldsymbol{\nu} + \rho_0^{-1}\boldsymbol{g})]\}(\boldsymbol{r},t)\phi(\boldsymbol{r}',t')\rangle = 0. \quad (42)$$

With the aid of identity (26), the Fourier transform of this equation becomes

$$\partial_t G_{\rho\phi}(\boldsymbol{k},t) + ik_\alpha G_{g_\alpha\phi}(\boldsymbol{k},t) = 0. \quad (43)$$

This equation implies that our SD equation preserves the mass conservation law. We also derive the identity by the substitution of (43) from (34).

$$F_{\hat{\rho}\psi}(\boldsymbol{k},t) = ik_\alpha T\rho_0 F_{\hat{\nu}_\alpha\psi}(\boldsymbol{k},t). \quad (44)$$

IV. PERTURBATION IN FIRST-LOOP ORDER

In this section, we develop the perturbative calculation of the SD equation within the first-loop order approximation. When we assume that the correlations including momentum are irrelevant in the long time limit, we can obtain an equation for the nonergodic parameter in the long time limit.

From Eqs. (36), (39), (40), and (43), we thus obtain the time evolution of the density correlation function as

$$\partial_t^2 G_{\rho\rho}(\boldsymbol{k},t) + \rho_0^{-1}L\partial_t G_{\rho\rho}(\boldsymbol{k},t) + T\rho_0 k^2 K(\boldsymbol{k})G_{\rho\rho}(\boldsymbol{k},t) = -T\rho_0 k^2 F_{\hat{\theta}\rho}(\boldsymbol{k},t) - ik_\alpha F_{\hat{g}_\alpha\rho}(\boldsymbol{k},t) + iTLk_\alpha F_{\hat{\nu}_\alpha\rho}(\boldsymbol{k},t), \quad (45)$$

where $L \equiv \delta_{\alpha\beta}L_{\alpha\beta}$. It is remarkable that the left-hand side (LHS) of Eq. (45) is equivalent to the SMCT without memory functions when we omit the terms that include the self-energies. However, this equation is quite complicated, because the self-energies are included in $F_{\hat{\theta}\rho}$, $F_{\hat{g}_\alpha\rho}$, and $F_{\hat{\nu}_\alpha\rho}$. Therefore, we restrict our interest to the calculation of the self-energies in the first-loop order perturbation in the latter part of this paper.

In the first-loop order perturbation, the self-energy $\Sigma_{\hat{\phi}_1\hat{\phi}_1'}$ is expressed as

$$\Sigma_{\hat{\phi}_1\hat{\phi}_1'}(X_1,X_1') = \frac{1}{2}\sum_{\phi_2,\phi_3,\phi_2',\phi_3'}\int dX_2dX_3dX_2'dX_3' \times V_{\hat{\phi}_1\phi_2\phi_3}(X_1,X_2,X_3)V_{\hat{\phi}_1'\phi_2'\phi_3'}(X_1',X_2',X_3') \times G_{\phi_2\phi_2'}(X_2-X_2')G_{\phi_3\phi_3'}(X_3-X_3'), \quad (46)$$

where $\hat{\phi}_1$ or $\hat{\phi}_1'$ is one of the components of $\hat{\phi}$, and ϕ_i or ϕ_i' is one of the components of ϕ. Here, the three-point vertex $V_{\hat{\phi}_1\phi_2\phi_3}$ is defined by

$$V_{\hat{\phi}_1\phi_2\phi_3}(X_1,X_2,X_3) \equiv \frac{\delta^3 S_{ng}[\phi]}{\delta\phi_1(X_1)\delta\phi_2(X_2)\delta\phi_3(X_3)}. \quad (47)$$

We list all three-point vertices in Appendix D. Note that there are no four-point correlation functions including both $\hat{\phi}_1$ and $\hat{\phi}_1'$.

Within the first-loop order approximation, $F_{\hat{\phi}\phi}$ in Eq. (38) is reduced to

$$F_{\hat{\phi}\phi}(\boldsymbol{k},t) \simeq \int_0^t ds(\Sigma_{\hat{\phi}\hat{\theta}}(\boldsymbol{k},t-s)\partial_s G_{\rho\phi}(\boldsymbol{k},s) - (T\rho_0)^{-1}\Sigma_{\hat{\phi}\hat{g}_\alpha}(\boldsymbol{k},t-s)G_{g_\alpha\phi}(\boldsymbol{k},s) - \Sigma_{\hat{\phi}\hat{g}_\alpha}(\boldsymbol{k},t-s)F_{\hat{\nu}_\alpha\phi}(\boldsymbol{k},s) + \Sigma_{\hat{\phi}\hat{\nu}_\alpha}(\boldsymbol{k},t-s)\{\partial_s G_{g_\alpha\phi}(\boldsymbol{k},s) + ik_\alpha T\rho_0[K(\boldsymbol{k})G_{\rho\phi}(\boldsymbol{k},s) + G_{\theta\phi}(\boldsymbol{k},s)]\})$$
$$\simeq \int_0^t ds[\Sigma_{\hat{\phi}\hat{\theta}}(\boldsymbol{k},t-s)\partial_s G_{\rho\phi}(\boldsymbol{k},s) - (T\rho_0)^{-1}\Sigma_{\hat{\phi}\hat{g}_\alpha}(\boldsymbol{k},t-s)G_{g_\alpha\phi}(\boldsymbol{k},s) - \Sigma_{\hat{\phi}\hat{\nu}_\alpha}(\boldsymbol{k},t-s)\rho_0^{-1}L_{\alpha\beta}G_{g_\beta\phi}(\boldsymbol{k},s)]. \quad (48)$$

We have used Eqs. (36), (39), and (40) and eliminated higher order contributions to obtain the final expression. We have also used Eq. (40) and the relation

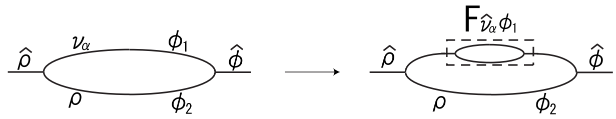

FIG. 1. A diagram of the self-energy $\Sigma_{\hat{\rho}\hat{\phi}}$ produced by vertex $V_{\hat{\rho}\rho\nu_\alpha}$. From Eq. (40), the first-loop order self-energy on the left-hand side of the figure can be treated as a second-loop order one as shown on the right-hand side of the figure.

$$\Sigma_{\hat{\phi}\hat{\rho}}(\boldsymbol{k},t) \simeq -ik_\alpha T\rho_0 \Sigma_{\hat{\phi}\hat{\nu}_\alpha}(\boldsymbol{k},t), \tag{49}$$

which is only valid in the first-loop order, for the first equality in Eq. (48). Let us show expression (49). The self-energy $\Sigma_{\hat{\phi}\hat{\rho}}$ should contain the vertex $V_{\hat{\rho}\rho g_\alpha}$ or $V_{\hat{\rho}\rho\nu_\alpha}$. Among these two vertices, vertex $V_{\hat{\rho}\rho\nu_\alpha}$ is irrelevant within the first-loop order calculation. Indeed, the self-energy with vertex $V_{\hat{\rho}\rho\nu_\alpha}$ should contain the propagator $G_{\nu_\alpha\phi}$ that is equal to $F_{\nu_\alpha\phi}$ from Eq. (40) and is the first-loop order. Thus, eventually, the self-energy $\Sigma\hat{\phi}\hat{\rho}$ including $V_{\hat{\rho}\rho\nu_\alpha}$ becomes a higher order correction, as shown in Fig. 1. On the other hand, from Eqs. (D1) and (D9), the vertices $V_{\hat{\rho}\rho g_\beta}$ and $V_{\hat{\nu}_\alpha\rho g_\beta}$ satisfy the relation

$$V_{\hat{\rho}\rho g_\beta}(X_1,X_2,X_3) = T\rho_0 \nabla_{\boldsymbol{r}_1\alpha} V_{\hat{\nu}_\alpha\rho g_\beta}(X_1,X_2,X_3). \tag{50}$$

Thus, we obtain relation (49).

Let us calculate some typical terms, such as $F_{\hat{\theta}\rho}(\boldsymbol{k},t)$, that appear on the RHS of Eq. (45) in the long time limit under the first-loop order approximation. For simplicity, we assume that the correlations including momentum to be negligible in the long time region. For this purpose, first, we calculate $\Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},t)$. Among the three-point vertex functions listed in Appendix D, there are only two vertices, (D3) and (D4), that include $\hat{\theta}$. Substituting (D3) and (D4) into (46) with $\hat{\phi}_1 = \hat{\phi}_1' = \hat{\theta}$, the expression of $\Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},t)$ at the first-loop order is given by

$$\begin{aligned}\Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},t) = \int \frac{d\boldsymbol{q}}{(2\pi)^3}\Bigg(&\frac{1}{2m^2\rho_0^4} G_{\rho\rho}(\boldsymbol{q},t)G_{\rho\rho}(\boldsymbol{k}-\boldsymbol{q},t) \\ &+ \frac{1}{Tm\rho_0^4}[G_{g_\alpha\rho}(\boldsymbol{q},t)G_{g_\alpha\rho}(\boldsymbol{k}-\boldsymbol{q},t) \\ &+ G_{\rho g_\alpha}(\boldsymbol{q},t)G_{\rho g_\alpha}(\boldsymbol{k}-\boldsymbol{q},t)] \\ &+ \frac{2}{T^2\rho_0^4} G_{g_\alpha g_\beta}(\boldsymbol{q},t)G_{g_\alpha g_\beta}(\boldsymbol{k}-\boldsymbol{q},t)\Bigg).\end{aligned} \tag{51}$$

Thus, in the limit $t\to\infty$, the first term of $F_{\hat{\theta}\rho}(\boldsymbol{k},t)$ in Eq. (48) can be approximated by

$$\begin{aligned}&\int_0^t ds \Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},t-s)\partial_s G_{\rho\rho}(\boldsymbol{k},s) \\ &\simeq \Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},t)\int_0^t ds\partial_s G_{\rho\rho}(\boldsymbol{k},s) \\ &\simeq \frac{G_{\rho\rho}(\boldsymbol{k},t)-G_{\rho\rho}(\boldsymbol{k},0)}{2m^2\rho_0^4}\int \frac{d\boldsymbol{q}}{(2\pi)^3}G_{\rho\rho}(\boldsymbol{q},t)G_{\rho\rho}(\boldsymbol{k}-\boldsymbol{q},t).\end{aligned} \tag{52}$$

Here, the last expression is obtained from the assumption that the correlations including momentum are irrelevant in the long time limit.

Similarly, with the aid of (46) and (D3)–(D8), $\Sigma_{\hat{\theta}\hat{g}_\alpha}(\boldsymbol{k},t)$ at the first-loop order calculation reduces to

$$\begin{aligned}\Sigma_{\hat{\theta}\hat{g}_\alpha}(\boldsymbol{k},t) &\simeq -\int \frac{d\boldsymbol{q}}{(2\pi)^3}\frac{iT}{m\rho_0^2}q_\alpha[K(\boldsymbol{q})G_{\rho\rho}(\boldsymbol{q},t)+G_{\theta\rho}(\boldsymbol{q},t)]G_{\rho\rho}(\boldsymbol{k}-\boldsymbol{q},t) \\ &= -\frac{iT}{m\rho_0^2}\int \frac{d\boldsymbol{q}}{(2\pi)^3}q_\alpha[K(\boldsymbol{q})G_{\rho\rho}(\boldsymbol{q},t)+F_{\hat{\theta}\rho}(\boldsymbol{q},t)]G_{\rho\rho}(\boldsymbol{k}-\boldsymbol{q},t) \\ &\simeq -\frac{iT}{m\rho_0^2}\int \frac{d\boldsymbol{q}}{(2\pi)^3}q_\alpha K(\boldsymbol{q})G_{\rho\rho}(\boldsymbol{q},t)G_{\rho\rho}(\boldsymbol{k}-\boldsymbol{q},t) \\ &= -\frac{iTk_\alpha}{mk^2\rho_0^2}\int \frac{d\boldsymbol{q}}{(2\pi)^3}k_\beta q_\beta K(\boldsymbol{q})G_{\rho\rho}(\boldsymbol{q},t)G_{\rho\rho}(\boldsymbol{k}-\boldsymbol{q},t),\end{aligned} \tag{53}$$

in the limit $t\to\infty$. The first equality in (53) originates from the assumption that the correlations including momentum are irrelevant in the long time limit. For the second equality in (53), we have used Eq. (39). To obtain the third equality in (53), we have ignored the contribution from $F_{\hat{\theta}\rho}$. This simplification can be justified at the first-loop order approximation, because $F_{\hat{\theta}\rho}$ is the first or above loop order function. To obtain the last expression in (53), we have used the fact that the density correlation function depends on time and the absolute value of the wave vector. From Eq. (53), the second term of $F_{\hat{\theta}\rho}(\boldsymbol{k},t)$ in Eq. (48) becomes

$$-\frac{1}{T\rho_0}\int_0^t ds \Sigma_{\hat{\theta}\hat{g}_\alpha}(\boldsymbol{k},t-s)G_{g_\alpha\rho}(\boldsymbol{k},s) \simeq \frac{i}{mk^2\rho_0^3}\int \frac{d\boldsymbol{q}}{(2\pi)^3}k_\alpha q_\alpha K(\boldsymbol{q})G_{\rho\rho}(\boldsymbol{q},t)G_{\rho\rho}(\boldsymbol{k}-\boldsymbol{q},t)\int_0^t ds k_\beta G_{g_\beta\rho}(\boldsymbol{k},s)$$
$$= \frac{-1}{mk^2\rho_0^3}\int \frac{d\boldsymbol{q}}{(2\pi)^3}k_\alpha q_\alpha K(\boldsymbol{q})G_{\rho\rho}(\boldsymbol{q},t)G_{\rho\rho}(\boldsymbol{k}-\boldsymbol{q},t)\int_0^t ds \partial_s G_{\rho\rho}(\boldsymbol{k},s)$$
$$= -\frac{G_{\rho\rho}(\boldsymbol{k},t)-G_{\rho\rho}(\boldsymbol{k},0)}{mk^2\rho_0^3}\int \frac{d\boldsymbol{q}}{(2\pi)^3}k_\alpha q_\alpha K(\boldsymbol{q})G_{\rho\rho}(\boldsymbol{q},t)G_{\rho\rho}(\boldsymbol{k}-\boldsymbol{q},t), \qquad (54)$$

where we have used Eq. (43) in the second equality. The last term of $F_{\hat{\theta}\rho}$ in Eq. (48) is zero because $\Sigma_{\hat{\theta}\hat{v}_\alpha}(\boldsymbol{k},t)$ and $G_{g_\beta\rho}(\boldsymbol{k},t)$ are zero in the long time limit.

Thus, we obtain the expression for $F_{\hat{\theta}\rho}(\boldsymbol{k},t)$ in Eq. (48) from Eqs. (52) and (54) at the first-loop order as

$$F_{\hat{\theta}\rho}(\boldsymbol{k},t) = \int \frac{d\boldsymbol{q}}{(2\pi)^3}\left(\frac{1}{2m^2\rho_0^4} - \frac{1}{mk^2\rho_0^3}k_\alpha q_\alpha K(\boldsymbol{q})\right) \times G_{\rho\rho}(\boldsymbol{q},t)G_{\rho\rho}(\boldsymbol{k}-\boldsymbol{q},t)[G_{\rho\rho}(\boldsymbol{k},t)-G_{\rho\rho}(\boldsymbol{k},0)], \qquad (55)$$

in the limit $t\to\infty$. Similarly, we evaluate $ik_\alpha F_{\hat{g}_\alpha\rho}(\boldsymbol{k},t)$ and $F_{\hat{v}_\alpha\rho}(\boldsymbol{k},t)$ within the first-loop order as

$$ik_\alpha F_{\hat{g}_\alpha\rho}(\boldsymbol{k},t) = \int \frac{d\boldsymbol{q}}{(2\pi)^3}\Bigg(-\frac{T}{m\rho_0^2}k_\alpha q_\alpha K(\boldsymbol{q}) + \frac{T}{2k^2\rho_0}[k_\alpha q_\alpha K(\boldsymbol{q}) + k_\alpha(k_\alpha - q_\alpha)K(\boldsymbol{k}-\boldsymbol{q})]^2\Bigg) \times G_{\rho\rho}(\boldsymbol{q},t)G_{\rho\rho}(\boldsymbol{k}-\boldsymbol{q},t)[G_{\rho\rho}(\boldsymbol{k},t)-G_{\rho\rho}(\boldsymbol{k},0)], \qquad (56)$$

and

$$F_{\hat{v}_\alpha\rho}(\boldsymbol{k},t) = 0, \qquad (57)$$

in the limit $t\to\infty$.

From the above expressions for $F_{\hat{\theta}\rho}(\boldsymbol{k},t)$, $F_{\hat{g}_\alpha\rho}(\boldsymbol{k},t)$, and $F_{\hat{v}_\alpha\rho}(\boldsymbol{k},t)$, we can evaluate the RHS of Eq. (45) in the limit $t\to\infty$. We note that the first and second terms on the LHS of Eq. (45) are zero in the long time limit, because the time derivatives of the density correlation functions should be zero in such a region. Therefore, the self-consistent equation of the nonergodic parameter $f(\boldsymbol{k})$, which is defined by

$$f(\boldsymbol{k}) \equiv \lim_{t\to\infty}\frac{G_{\rho\rho}(\boldsymbol{k},t)}{G_{\rho\rho}(\boldsymbol{k},0)}, \qquad (58)$$

is obtained as

$$f(\boldsymbol{k}) = \frac{M(\boldsymbol{k})}{1+M(\boldsymbol{k})}, \qquad (59)$$

where

$$M(\boldsymbol{k}) \equiv \frac{\rho_0 S(\boldsymbol{k})}{2mk^4}\int \frac{d\boldsymbol{q}}{(2\pi)^3}V_{\boldsymbol{k},\boldsymbol{q}}^2 S(\boldsymbol{p})S(\boldsymbol{q})f(\boldsymbol{p})f(\boldsymbol{q}), \qquad (60)$$

with

$$V_{\boldsymbol{k},\boldsymbol{q}} \equiv k_\alpha[q_\alpha c(\boldsymbol{q}) + p_\alpha c(\boldsymbol{p})], \qquad (61)$$

where we have used $\boldsymbol{p}\equiv\boldsymbol{k}-\boldsymbol{q}$ and the static structure factor (32). This set of self-consistent equations (58)–(61) for the nonergodic parameter is equivalent to that in the SMCT.

V. DISCUSSION AND CONCLUSION

A. Discussion

In this paper, we formulated the FDR-preserving field theory for FNH. The SMCT-like equation obtained under the first-loop order approximation in the preceding section might be the first step toward constructing a correct theory beyond the SMCT when higher-order perturbations are included. However, some issues remain to be clarified with regard to the analysis. Let us discuss these issues through comparison with other field theoretical approaches.

To analyze the SD equation, we considered the important assumption that the correlations including momentum are negligibly small in the long time region. This assumption is crucial to discuss whether ENE transition exists. Indeed, if we assume that the contributions from momentum are negligible, FNH equations are reduced to the Dean-Kawasaki equation, as indicated by Kawasaki [11]. On the other hand, there are several indirect evidences for the justification of this approximation. First, we note that a numerical simulation exhibits fast relaxations of the correlations including momentum [37]. Second, it is known that the density-density correlation $G_{\rho\rho}$ is connected to the correlation in the longitudinal mode $k_\alpha G_{g_\alpha\phi}$, which is proportional to $\partial_t G_{\rho\phi}$. Since the system is almost stationary, any terms including the time derivative are small. Thus, the contributions from the correlation including momentum can be ignored in the slow dynamics in the motion of the density field.

Let us compare our results with those obtained in other field theoretical researches, taking into account the following four aspects. (i) Is the basic model adequate? (ii) Is the analysis FDR preserved? (iii) Are the approximations to analyze the density correlation function valid in the limit $t\to\infty$? (iv) What is the behavior of the density correlation function in the limit $t\to\infty$? The results of the comparison are summarized in Table I.

TABLE I. The comparison of our results with Das and Mazenko (DM) [8], SDD [10], and ABL [23] and Kim and Kawasaki (KK) [28]. We list the results on the four points: (i) the used model, (ii) whether FDR is preserved in the perturbation, (iii) the used approximation to derive the density correlation function in the limit $t \to \infty$, and (iv) the behavior of the nonergodic parameter $f(k)$. Here, the expression $G_{g\phi} \to 0$ means that the correlations including momentum become zero in the long time limit. $f_{\mathrm{SMCT}}(k)$ means the nonergodic parameter is equivalent to that of SMCT.

Reference	(i) Model	(ii) FDR	(iii) Approximations	(iv)Nonergodic parameter
DM [8]	FNH	Nonpreserve	Nonperturbative	0
SDD [10]	FNH with violation of Galilean invariance	Preserve	first-loop order, $G_{g\phi} \to 0$	0
ABL [25]	Dean-Kawasaki and FNH	Preserve	first-loop order, $G_{g\phi} \to 0$	1
KK [32]	Dean-Kawasaki	Preserve	First-loop order	$f_{\mathrm{SMCT}}(k)$
This work	FNH	Preserve	First-loop order, $G_{g\phi} \to 0$	$f_{\mathrm{SMCT}}(k)$

First, we compare our approach with that of ABL [23]. As can be seen in Table I, ABL predicted the nonergodic parameter to be unity, a value independent of the wave number. This result is clearly in contrast to the observations in experiments and simulations. It might be caused by the definition of the new set of additional fields. Indeed, we have introduced θ in Eq. (15) and $\boldsymbol{\nu}$ in eq. (16), but ABL [23] used

$$\theta_{ABL} \equiv \frac{\delta F}{\delta \rho} \quad \text{and} \quad \nu_{\alpha,ABL} \equiv \frac{\partial F}{\partial g_\alpha}. \tag{62}$$

These new variables include the linear terms of $\delta\rho$ and $\boldsymbol{g}$, but our additional variables, θ and $\boldsymbol{\nu}$, do not include the linear terms. As a result, the order of correlations, which include the new variables θ and $\boldsymbol{\nu}$, differ from ours. Indeed, $G_{\theta\phi}$ and $G_{\nu_\alpha\phi}$ are the first or above loop orders in Eq. (39) and Eq. (40), while $G_{\theta\phi}$ and $G_{\nu_\alpha\phi}$ include the tree diagrams in the calculations of ABL. Therefore, we believe that Eq. (62) is not appropriate, and that Eqs. (15) and (16) should be used instead.

Next, we compare our results with those of Kim and Kawasaki [28]. Their method is almost parallel to the one we used. However, their basic equation is not the FNH equations but the Dean-Kawasaki equation. Thus, their MCT equation without interactions is the diffusion equation. On the other hand, our MCT equation (45) without a memory kernel is an equation for a damped oscillator. Thus, the existence of the momentum conservation equation in the basic equation naturally leads to the existence of the acceleration term in the MCT equation.

Third, let us compare our results with those of SDD [10], who used a simplified model of FNH. Although the approximation used in this study is similar to that used by SDD, the Galilean invariance is not preserved in their model equation. This implies that the violation of the conservation law cause the artificial cutoff mechanism.

Fourth, we compare our results with those of Das and Mazenko [8]. One important difference between the two results is that Das and Mazenko consider $\boldsymbol{V}$, which satisfies the constraint $\boldsymbol{V} \equiv \boldsymbol{g}/\rho$, as one collective variable. On the other hand, we introduce the new field variables, θ in Eq. (15) and $\boldsymbol{\nu}$ in Eq. (16), to satisfy FDR. Thus, their explicit expressions differ from ours. Second, let us discuss their conclusion on the existence of the cutoff mechanism, i.e., $G_{\rho\rho} \propto G_{\rho\hat{\rho}}=0$ in the long time limit. As indicated by ABL [23], the relation $G_{\rho\rho} \propto G_{\rho\hat{\rho}}$, used by Das and Mazenko [see Eq. (6.62) in [8]], does not preserve FDR. Thus, we cannot conclude $G_{\rho\rho}=0$ from the relation $G_{\rho\hat{\rho}}=0$. However, the relation $G_{\rho\hat{\rho}}=0$, which was derived from their nonperturbative analysis, might be valid. On the other hand, from Eq. (A5), our FDR-preserving calculation under the first-loop order perturbation suggests $G_{\rho\hat{\rho}}=K(\boldsymbol{k})G_{\rho\rho}+G_{\theta\rho} \neq 0$, for which we used the numerical result of the nonergodic parameter. Thus, our result in the first-loop order perturbation on $G_{\rho\hat{\rho}}$ is not consistent with that of Das and Mazenko. To resolve this discrepancy between their theory [8] and ours, or to verify their analysis of the cutoff mechanism, we need to find some identical relations without using approximations.

B. Conclusion

In this paper, we reformulate a FDR-preserving field theory starting from FNH. By assuming that the correlations including momentum are irrelevant in the long time region, we have shown that the nonergodic parameter under the first-loop order approximation satisfies an equation that is equivalent to the SMCT in the long time limit. Thus, we believe that by analyzing correlations in higher-loop orders, we will be able to construct a correct theory to explain the experimental and numerical results.

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APPENDIX A: TIME-REVERSAL TRANSFORMATION AND FDR

From the linearity of the time-reversal transformation, we can rewrite Eq. (21) as follows:

$$\begin{pmatrix} \rho \\ \hat{\rho} \\ \theta \\ \hat{\theta} \\ g_\alpha \\ \hat{g}_\alpha \\ \nu_\alpha \\ \hat{\nu}_\alpha \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \partial_t & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\delta_{\alpha\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{T\rho_0}\delta_{\alpha\beta} & \delta_{\alpha\beta} & -\delta_{\alpha\beta} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\delta_{\alpha\beta} & 0 \\ 0 & 0 & 0 & 0 & -\delta_{\alpha\beta}\partial_t & 0 & 0 & -\delta_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \rho \\ \hat{\rho} \\ \theta \\ \hat{\theta} \\ g_\beta \\ \hat{g}_\beta \\ \nu_\beta \\ \hat{\nu}_\beta \end{pmatrix}. \tag{A1}$$

When we express this time-reversal transformation matrix as O, Eq. (21) can be represented by $\psi \to \mathsf{O}\psi$. This implies that the correlation function satisfies

$$\mathsf{G}(-t) = \mathsf{O}\mathsf{G}(t)\mathsf{O}^T. \tag{A2}$$

Similarly, the self-energy satisfies

$$\Sigma(-t) = (\mathsf{O}^T)^{-1}\Sigma(t)\mathsf{O}^{-1}, \tag{A3}$$

where we have used relation (30) or

$$\mathsf{G}^{-1} = \mathsf{G}_0^{-1} - \Sigma. \tag{A4}$$

Here, we avoid writing all components of the time-reversal symmetry relations (A2) and (A3), because they are very long. Instead, we present some typical relations, which are necessary for the calculation of the $\hat{\theta}\rho$ component of the SD equation,

$$G_{\hat{\rho}\rho}(\boldsymbol{k},t) = \Theta(-t)[K(\boldsymbol{k})G_{\rho\rho}(\boldsymbol{k},t) + G_{\theta\rho}(\boldsymbol{k},t)], \tag{A5}$$

$$G_{\hat{\theta}\rho}(\boldsymbol{k},t) = -\Theta(-t)\partial_t G_{\rho\rho}(\boldsymbol{k},t), \tag{A6}$$

$$G_{\hat{g}_\alpha\rho}(\boldsymbol{k},t) = \Theta(-t)\left(\frac{1}{T\rho_0}G_{g_\alpha\rho}(\boldsymbol{k},t) + G_{\hat{\nu}_\alpha\rho}(\boldsymbol{k},t)\right), \tag{A7}$$

$$G_{\hat{\nu}_\alpha\rho}(\boldsymbol{k},t) = -\Theta(-t)\partial_t G_{g_\alpha\rho}(\boldsymbol{k},t). \tag{A8}$$

Here, we summarize some relevant relations among the self-energies

$$\Sigma_{\hat{\theta}\rho}(\boldsymbol{k},t) = \Theta(t)(\partial_t\Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},t) - K(\boldsymbol{k})\Sigma_{\hat{\theta}\hat{\rho}}(\boldsymbol{k},t)), \tag{A9}$$

$$\Sigma_{\hat{\theta}\theta}(\boldsymbol{k},t) = -\Theta(t)\Sigma_{\hat{\theta}\hat{\rho}}(\boldsymbol{k},t), \tag{A10}$$

$$\Sigma_{\hat{\theta}g_\alpha}(\boldsymbol{k},t) = -\Theta(t)\left(\frac{1}{T\rho_0}\Sigma_{\hat{\theta}\hat{g}_\alpha}(\boldsymbol{k},t) - \partial_t\Sigma_{\hat{\theta}\hat{\nu}_\alpha}(\boldsymbol{k},t)\right), \tag{A11}$$

$$\Sigma_{\hat{\theta}\nu_\alpha}(\boldsymbol{k},t) = -\Theta(t)\Sigma_{\hat{\theta}\hat{g}_\alpha}(\boldsymbol{k},t), \tag{A12}$$

$$\Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},t) = \Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},-t), \tag{A13}$$

$$\Sigma_{\hat{\theta}\hat{\nu}_\alpha}(\boldsymbol{k},t) = -\Sigma_{\hat{\theta}\hat{\nu}_\alpha}(\boldsymbol{k},-t). \tag{A14}$$

APPENDIX B: CALCULATION OF FDR-PRESERVING SD EQUATION

In this appendix, we calculate the $\hat{\theta}\rho$ component of the SD equation with the aid of the result in Appendix A. From Eqs. (23) and (31), only the $-\hat{\theta}\theta$ term exists in S_g, which includes $\hat{\theta}$. Therefore, we obtain $(G_0^{-1})_{\hat{\theta}\phi}(X_1-X_2) = \delta_{\phi\theta}\delta(X_1-X_2)$. Thus, the $\hat{\theta}\rho$ component of $\mathsf{G}_0{}^{-1}\cdot\mathsf{G}$ satisfies

$$[G_0{}^{-1}\cdot\mathsf{G}]_{\hat{\theta}\rho}(\boldsymbol{k},t) = G_{\theta\rho}(\boldsymbol{k},t). \tag{B1}$$

On the other hand, the $\hat{\theta}\rho$ component of $\Sigma\cdot\mathsf{G}$ is expressed as

$$\begin{aligned}[\Sigma\cdot\mathsf{G}]_{\hat{\theta}\rho}(\boldsymbol{k},t) = \int_{-\infty}^{\infty} ds[&\Sigma_{\hat{\theta}\rho}(\boldsymbol{k},t-s)G_{\rho\rho}(\boldsymbol{k},s) + \Sigma_{\hat{\theta}\theta}(\boldsymbol{k},t-s)G_{\theta\rho}(\boldsymbol{k},s) + \Sigma_{\hat{\theta}g_\alpha}(\boldsymbol{k},t-s)G_{g_\alpha\rho}(\boldsymbol{k},s) + \Sigma_{\hat{\theta}\nu_\alpha}(\boldsymbol{k},t-s)G_{\nu_\alpha\rho}(\boldsymbol{k},s) \\ &+ \Sigma_{\hat{\theta}\hat{\rho}}(\boldsymbol{k},t-s)G_{\hat{\rho}\rho}(\boldsymbol{k},s) + \Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},t-s)G_{\hat{\theta}\rho}(\boldsymbol{k},s) + \Sigma_{\hat{\theta}\hat{g}_\alpha}(\boldsymbol{k},t-s)G_{\hat{g}_\alpha\rho}(\boldsymbol{k},s) + \Sigma_{\hat{\theta}\hat{\nu}_\alpha}(\boldsymbol{k},t-s)G_{\hat{\nu}_\alpha\rho}(\boldsymbol{k},s)].\end{aligned} \tag{B2}$$

By using relations (A9)–(A12), the first four terms on the right-hand side of Eq. (B2) can be rewritten as

$$-\int_{-\infty}^{t} ds\Bigg[[\partial_s\Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},t-s)+K(\boldsymbol{k})\Sigma_{\hat{\theta}\hat{\rho}}(\boldsymbol{k},t-s)]G_{\rho\rho}(\boldsymbol{k},s)+\Sigma_{\hat{\theta}\hat{\rho}}(\boldsymbol{k},t-s)G_{\theta\rho}(\boldsymbol{k},s)$$
$$+\left(\frac{1}{T\rho_0}\Sigma_{\hat{\theta}\hat{g}_\alpha}(\boldsymbol{k},t-s)+\partial_s\Sigma_{\hat{\theta}\hat{\nu}_\alpha}(\boldsymbol{k},t-s)\right)G_{g_\alpha\rho}(\boldsymbol{k},s)+\Sigma_{\hat{\theta}\hat{g}_\alpha}(\boldsymbol{k},t-s)G_{\nu_\alpha\rho}(\boldsymbol{k},s)\Bigg]$$
$$=-\Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},0)G_{\rho\rho}(\boldsymbol{k},t)-\int_{-\infty}^{t} ds\Bigg[\Sigma_{\hat{\theta}\hat{\rho}}(\boldsymbol{k},t-s)[K(\boldsymbol{k})G_{\rho\rho}(\boldsymbol{k},s)+G_{\theta\rho}(\boldsymbol{k},s)]-\Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},t-s)\partial_s G_{\rho\rho}(\boldsymbol{k},s)$$
$$+\Sigma_{\hat{\theta}\hat{g}_\alpha}(\boldsymbol{k},t-s)\left(\frac{1}{T\rho_0}G_{g_\alpha\rho}(\boldsymbol{k},s)+G_{\nu_\alpha\rho}(\boldsymbol{k},s)\right)-\Sigma_{\hat{\theta}\hat{\nu}_\alpha}(\boldsymbol{k},t-s)\partial_s G_{g_\alpha\rho}(\boldsymbol{k},s)\Bigg], \tag{B3}$$

where the boundary terms vanish except for $\Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},0)G_{\rho\rho}(\boldsymbol{k},t)$ because of $\Sigma_{\hat{\theta}\hat{\nu}_\alpha}(\boldsymbol{k},0)=0$ from (A14) and $G_{\rho\rho}(\boldsymbol{k},-\infty)=G_{g_\alpha\rho}(\boldsymbol{k},-\infty)=0$.

Similarly, from relations (A5)–(A8) the last four terms on the right-hand side of Eq. (B2) become

$$\int_{-\infty}^{0} ds\Bigg[\Sigma_{\hat{\theta}\hat{\rho}}(\boldsymbol{k},t-s)[K(\boldsymbol{k})G_{\rho\rho}(\boldsymbol{k},s)+G_{\theta\rho}(\boldsymbol{k},s)]$$
$$-\Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},t-s)\partial_s G_{\rho\rho}(\boldsymbol{k},s)$$
$$+\Sigma_{\hat{\theta}\hat{g}_\alpha}(\boldsymbol{k},t-s)\left(\frac{1}{T\rho_0}G_{g_\alpha\rho}(\boldsymbol{k},s)+G_{\nu_\alpha\rho}(\boldsymbol{k},s)\right)$$
$$-\Sigma_{\hat{\theta}\hat{\nu}_\alpha}(\boldsymbol{k},t-s)\partial_s G_{g_\alpha\rho}(\boldsymbol{k},s)\Bigg]. \tag{B4}$$

From Eqs. (B1)–(B4), we obtain the $\hat{\theta}\rho$ component of the FDR-preserving SD equation as

$$G_{\theta\rho}(\boldsymbol{k},t)+\Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},0)G_{\rho\rho}(\boldsymbol{k},t)$$
$$=-\int_{0}^{t} ds\Bigg[\Sigma_{\hat{g}_\beta\hat{\rho}}(\boldsymbol{k},t-s)[K(\boldsymbol{k})G_{\rho\rho}(\boldsymbol{k},s)+G_{\theta\rho}(\boldsymbol{k},s)]$$
$$-\Sigma_{\hat{g}_\beta\hat{\theta}}(\boldsymbol{k},t-s)\partial_s G_{\rho\rho}(\boldsymbol{k},s)-\Sigma_{\hat{g}_\beta\hat{\nu}_\alpha}(\boldsymbol{k},t-s)\partial_s G_{g_\alpha\rho}(\boldsymbol{k},s)$$
$$+\Sigma_{\hat{g}_\beta\hat{g}_\alpha}(\boldsymbol{k},t-s)\left(\frac{1}{T\rho_0}G_{g_\alpha\rho}(\boldsymbol{k},s)+G_{\nu_\alpha\rho}(\boldsymbol{k},s)\right)\Bigg]. \tag{B5}$$

Similarly, other components of the SD equation can be obtained with the aid of the time-reversal symmetry (A1).

APPENDIX C: SOME EXACT RELATIONS OF EQUAL-TIME CORRELATION FUNCTIONS AND SELF-ENERGIES

In this appendix, we derive some relations for the equal-time correlation functions and the self-energies from the effective free energy F. Here, we note that the mean value is calculated by the canonical average over F. Since the equal-time correlation function satisfies

$$\left\langle \delta\rho(\boldsymbol{r})\frac{\delta F}{\delta\rho(\boldsymbol{r}')}\right\rangle = T\delta(\boldsymbol{r}-\boldsymbol{r}')$$
$$= T\langle\delta\rho(\boldsymbol{r})K*\delta\rho(\boldsymbol{r}')\rangle + T\langle\delta\rho(\boldsymbol{r})\theta(\boldsymbol{r}')\rangle. \tag{C1}$$

The Fourier transform of this equation becomes

$$\langle\delta\rho(\boldsymbol{k})\theta(-\boldsymbol{k})\rangle=0, \tag{C2}$$

where we have used the relation $K(\boldsymbol{k})=G_{\rho\rho}^{-1}(\boldsymbol{k},0)$. Thus, we obtain

$$G_{\rho\theta}(\boldsymbol{k},0)=0. \tag{C3}$$

Substituting (C3) into Eq. (35), and setting $t=0$, we obtain

$$\Sigma_{\hat{\theta}\hat{\theta}}(\boldsymbol{k},0)=0. \tag{C4}$$

Similarly, from the relations

$$\left\langle g_\alpha(\boldsymbol{r})\rho(\boldsymbol{r}')\frac{\delta F}{\delta g_\beta(\boldsymbol{r}')}\right\rangle = T\rho_0\delta_{\alpha\beta}\delta(\boldsymbol{r}-\boldsymbol{r}')=\langle g_\alpha(\boldsymbol{r})g_\beta(\boldsymbol{r}')\rangle \tag{C5}$$

and

$$\left\langle g_\alpha(\boldsymbol{r})\frac{\delta F}{\delta g_\beta(\boldsymbol{r}')}\right\rangle = T\delta_{\alpha\beta}\delta(\boldsymbol{r}-\boldsymbol{r}')$$
$$=\rho_0^{-1}\langle g_\alpha(\boldsymbol{r})g_\beta(\boldsymbol{r}')\rangle + T\langle g_\alpha(\boldsymbol{r})\nu_\beta(\boldsymbol{r}')\rangle, \tag{C6}$$

we obtain

$$G_{g_\alpha\nu_\beta}(\boldsymbol{k},0)=G_{\nu_\beta g_\alpha}(\boldsymbol{k},0)=0. \tag{C7}$$

Substituting Eq. (C7), into Eq. (37) at $t=0$, we obtain

$$\Sigma_{\hat{\nu}_\alpha\hat{\nu}_\beta}(\boldsymbol{k},0)=0. \tag{C8}$$

APPENDIX D: LIST OF THREE-POINT VERTICES

In this appendix, we present all three-point vertices that are defined by (47),

$$V_{\hat{\rho}\rho g_\alpha}(X_1,X_2,X_3)=-\rho_0^{-1}\nabla_{r_1\alpha}\delta(X_1-X_2)\delta(X_1-X_3), \tag{D1}$$

$$V_{\hat{\rho}\rho\nu_\alpha}(X_1,X_2,X_3) = -T\nabla_{\boldsymbol{r}_1\alpha}\delta(X_1-X_2)\delta(X_1-X_3), \quad \text{(D2)}$$

$$V_{\hat{\theta}\rho\rho}(X_1,X_2,X_3) = -\frac{1}{m\rho_0^2}\delta(X_1-X_2)\delta(X_1-X_3), \quad \text{(D3)}$$

$$V_{\hat{\theta}g_\alpha g_\beta}(X_1,X_2,X_3) = -\frac{1}{T\rho_0^2}\delta_{\alpha\beta}\delta(X_1-X_2)\delta(X_1-X_3), \quad \text{(D4)}$$

$$V_{\hat{g}_\alpha\rho\rho}(X_1,X_2,X_3) = -T[\delta(X_1-X_2)\nabla_{\boldsymbol{r}_1\alpha}K(X_1-X_3) + \delta(X_1-X_3)\nabla_{\boldsymbol{r}_1\alpha}K(X_1-X_2)], \quad \text{(D5)}$$

$$V_{\hat{g}_\alpha\rho\theta}(X_1,X_2,X_3) = -T\delta(X_1-X_2)\nabla_{\boldsymbol{r}_1\alpha}\delta(X_1-X_3), \quad \text{(D6)}$$

$$V_{\hat{g}_\alpha g_\beta g_\gamma}(X_1,X_2,X_3) = -\rho_0^{-1}\{\delta_{\alpha\beta}\nabla_{\boldsymbol{r}_1\gamma}[\delta(X_1-X_2)\delta(X_1-X_3)] + \delta_{\beta\gamma}\delta(X_1-X_2)\nabla_{\boldsymbol{r}_1\alpha}\delta(X_1-X_3)\}, \quad \text{(D7)}$$

$$V_{\hat{g}_\alpha g_\beta \nu_\gamma}(X_1,X_2 . X_3) = -T\{\delta_{\alpha\beta}\nabla_{\boldsymbol{r}_1\gamma}[\delta(X_1-X_2)\delta(X_1-X_3)] + \delta_{\beta\gamma}\delta(X_1-X_2)\nabla_{\boldsymbol{r}_1\alpha}\delta(X_1-X_3)\}, \quad \text{(D8)}$$

$$V_{\hat{\nu}_\alpha\rho g_\beta}(X_1,X_2,X_3) = -\frac{1}{T\rho_0^2}\delta_{\alpha\beta}\delta(X_1-X_2)\delta(X_1-X_3), \quad \text{(D9)}$$

where $\delta(X_1-X_2)\equiv\delta(\boldsymbol{r}_1-\boldsymbol{r}_2,t_1-t_2)$ and $K(X_1-X_2)\equiv\delta(t_1-t_2)K*\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)$.

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